G1B

An Introduction to

Beam Dynamics

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Preface

Beam dynamics is the study of particle beams, their motion in environments, involving external electro-magnetic fields and their interactions, including the interaction of beams with matters, of beams with beams, and of particle beams with radiation. Evolving from concepts and ideas derived from classical mechanics, electromagnetism, statistical physics, and quantum physics. The study of beams is opening up a very rich field, with new effects being discovered and new types of beams with novel characteristics being realized. Basic knowledge of the beam physics is briefly introduced in this lecture for the students who are preparing to work in the field of Particle accelerators.

Contents

- Basic Concepts
- Transverse Motion
- Longitudinal Motion
- Collective Effects
- Lepton and Hadron

1. Basic Concepts

- Constants & Relations
- Motion in E-M Fields
- Linear accelerators
- Synchrotrons

1.1 Constants & Relations

- **Speed of light** $c = 2.99792458 \times 10^{10}$ **cm/sec**
- **Relativistic energy** $E = mc^2 = m_0 \gamma c^2$
- **Relativistic momentum** $p = mv = m_0 \gamma \beta c$

  $\beta = v/c \qquad \gamma = (1-\beta^2)^{-1/2}$
- ***E*-*p* relationship** $E^2/c^2 = p^2 + m_0^2 c^2$

  Ultra-relativistic particles: $\beta \approx 1$, $E \approx pc$
- **Kinetic energy** $T = E - m_0 c^2 = m_0 c^2 (\gamma - 1)$
- **Equation of motion under Lorentz force** $\dfrac{d\vec{p}}{dt} = \vec{f} \Rightarrow m_0 \dfrac{d}{dt}(\gamma \vec{v}) = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$

Constants & Relations *(cont.)*

- **Planck constant** $h = 6.626075 \times 10^{-34}$ J s
- **Electron charge** $e = 1.6021773 \times 10^{-19}$ Coulumbs
- **Electron volts** 1eV$= 1.6021773 \times 10^{-19}$ Joule
- **Energy in eV** $$E[\mathrm{eV}] = \frac{mc^2}{e} = \frac{m_0 \gamma c^2}{e}$$
- **Energy and rest mass** $1\mathrm{eV}/c^2 = 1.78 \times 10^{-36}$ kg
  - *Electron* $m_{0,\,e} = 0.51099906\ \mathrm{MeV}/c^2 = 9.1093897 \times 10^{-31}$ kg
  - *Proton* $m_{0,\,p} = 938.2723\ \mathrm{MeV}/c^2 = 1.6726231 \times 10^{-27}$ kg

1.2 Motion in E-M Fields

- **Governed by Lorentz force** $$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \vec{v} \times \vec{B}\right]$$

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

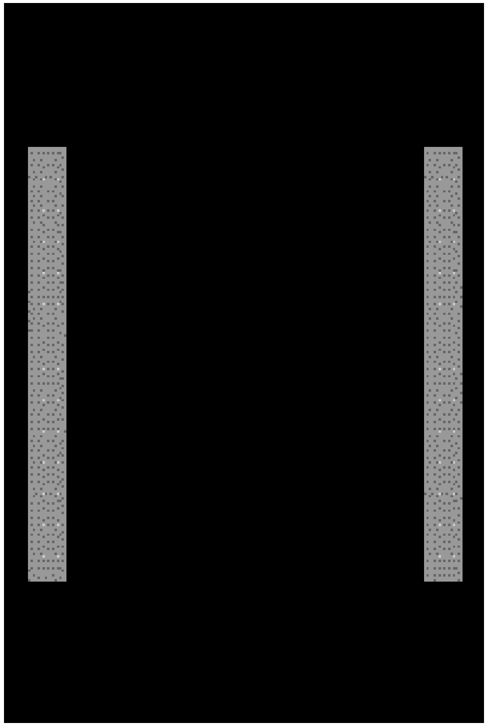
$$\Rightarrow \quad E\frac{dE}{dt} = c^2 \vec{p} \cdot \frac{d\vec{p}}{dt}$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{qc^2}{E}\vec{p} \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) = \frac{qc^2}{E}\vec{p} \cdot \vec{E}$$

A magnetic field does not alter a particle's energy. Only an electric field can do this.

- **Acceleration along a uniform electric field (*B*=0)**

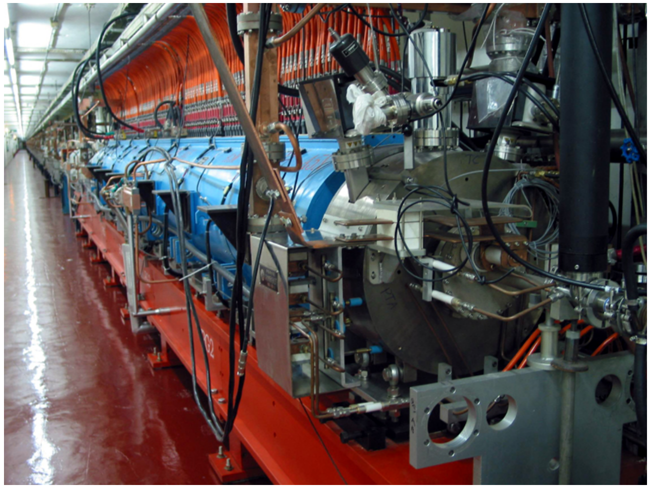
$$\left.\begin{aligned} z &\approx vt \\ x &\approx \frac{eE}{2m_0\gamma}t^2 \end{aligned}\right\} \quad \text{parabolic path for } v \ll c$$



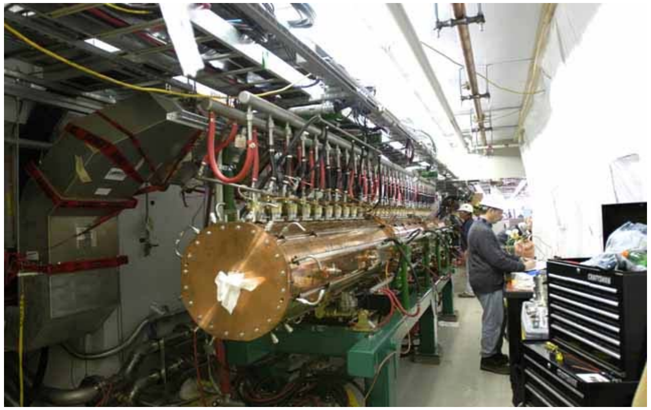
***C.R. Prior*: The Physics of Accelerators**

1.3 Linear Accelerators

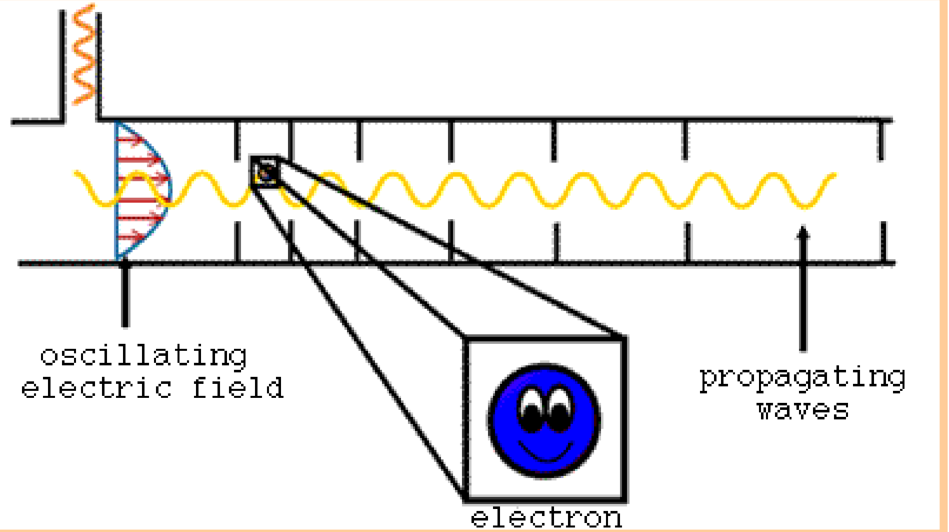
- **Simplest example is a vacuum chamber with one or more DC accelerating structures with the *E*-field aligned in the direction of motion.**
  - ***Limited to a few MeV***
- **To achieve energies higher than the highest voltage in the system, the *E*-fields are alternating at RF cavities.**
  - ***Avoids expensive magnets***
  - ***No loss of energy from synchrotron radiation (q.v.)***
  - ***But requires many structures, limited energy gain/metre***
  - ***Large energy increase requires a long accelerator***



BEPC electron-positron linac



SNS Linac, Oak Ridge

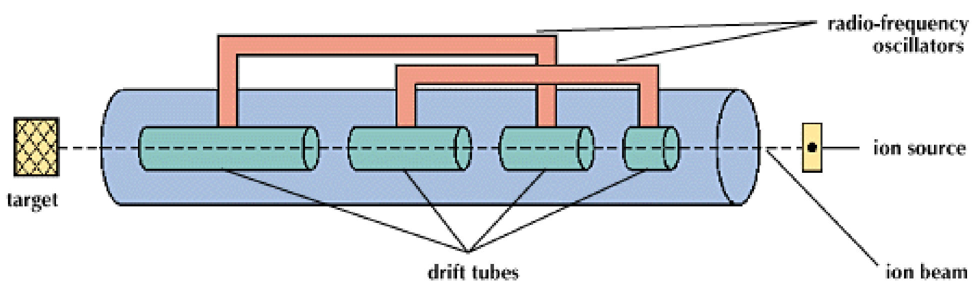


Structure 1:

- Travelling wave structure: particles keep in phase with the accelerating waveform.
- Phase velocity in the waveguide is greater than *c* and needs to be reduced to the particle velocity with a series of irises inside the tube whose polarity changes with time.
- In order to match the phase of the particles with the polarity of the irises, the distance between the irises increases farther down the structure where the particle is moving faster. But note that electrons at 3 MeV are already at 0.99c.

Structure 2:

- A series of drift tubes alternately connected to high frequency oscillator.
- Particles accelerated in gaps, drift inside tubes .
- For constant frequency generator, drift tubes increase in length as velocity increases.
- Beam has pulsed structure.



1.4 Synchrotrons

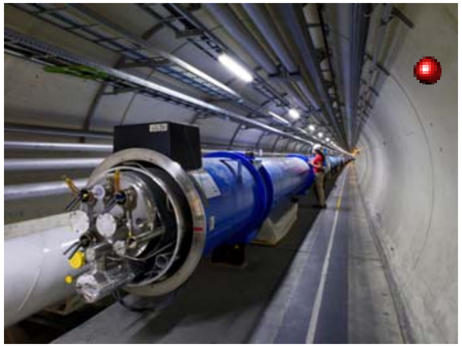
$$\rho = \left|\frac{p}{qB}\right|$$

$$f = n\omega$$

$$\omega = \frac{qBc^2}{E} = \frac{v}{\rho}$$

- **Principle of frequency modulation but in addition variation in time of *B*-field to match increase in energy and keep revolution radius constant.**
- **Magnetic field produced by several bending magnets (*dipoles*), increases linearly with momentum. For *q=e* and high energies:**

$$B\rho = \frac{p}{e} \approx \frac{E}{ce} \text{ so } E[\text{GeV}] \approx 0.3\, B[\text{T}]\, \rho[\text{m}]$$

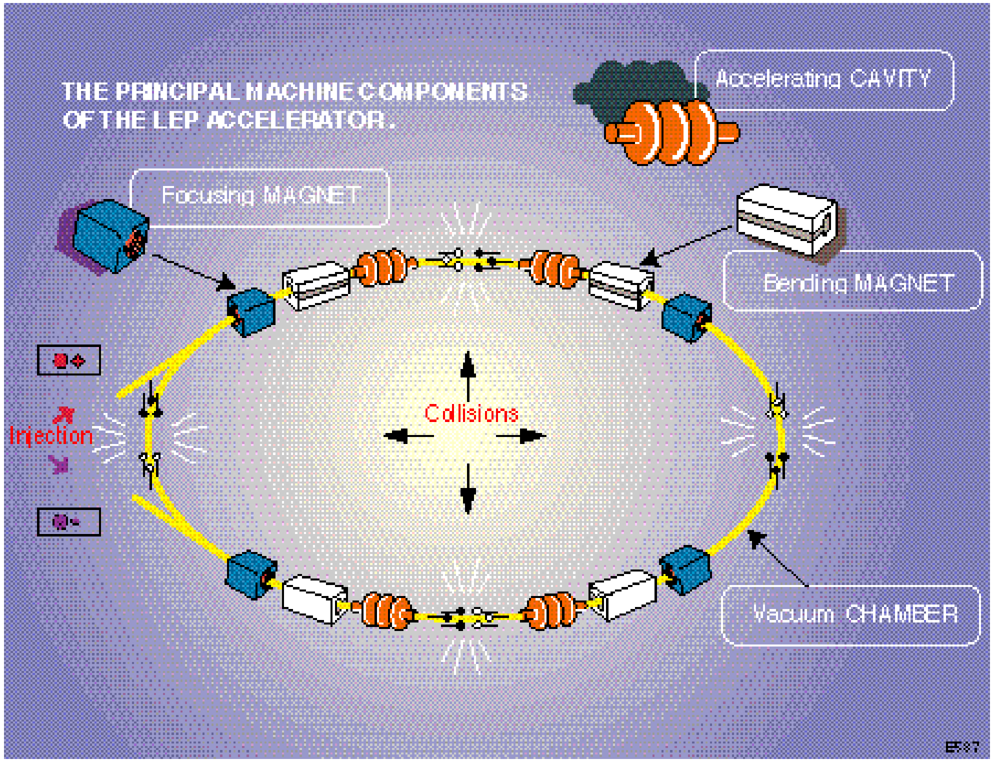


- **Practical limitations for magnetic fields => high energies only at large radius**

e.g. LHC: E = 7 TeV, B = 8.36 T, ρ = 2.7 km

Types of Synchrotrons

- Storage rings: accumulate particles and keep circulating for long periods; used for high intensity beams to inject into more powerful machines or synchrotron radiation factories.
- Colliders: two beams circulating in opposite directions, made to intersect; maximises energy in centre of mass frame.

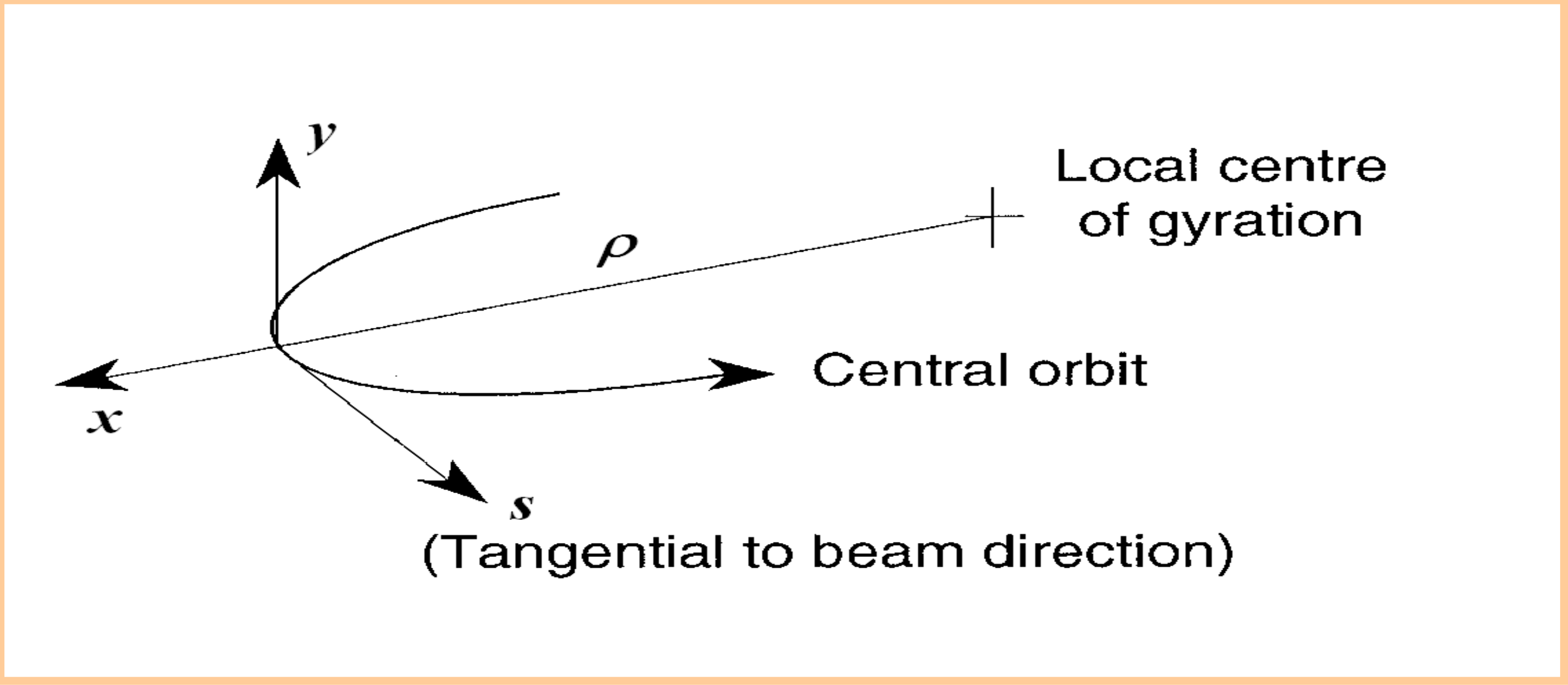


2. Transverse Motion

(Prof. S.Y.Lee)

- Motion Description
- Bending
- Focusing
- Hill's Equation
- Phase Space
- Lattice
- Dispersion
- Orbit Distortion
- Coupling
- Non-linearity

2.1 Motion Description



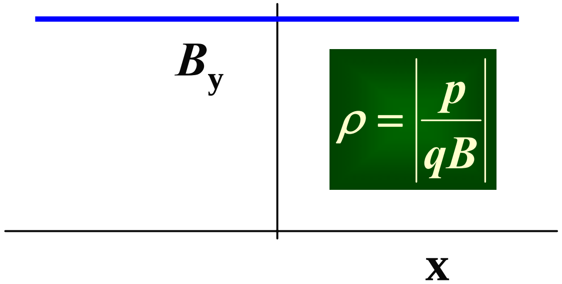
- *x*- horizontal
- *y*- vertical
- *s*- longitudinal

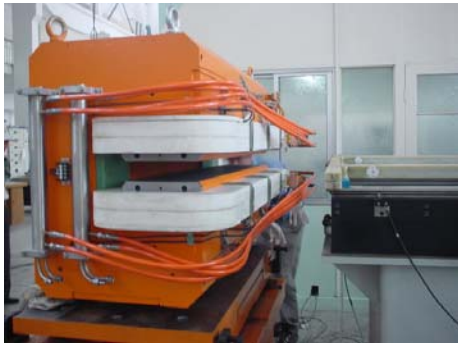
Transverse

2.2 Bending

(Prof. C.Ch. Wang)

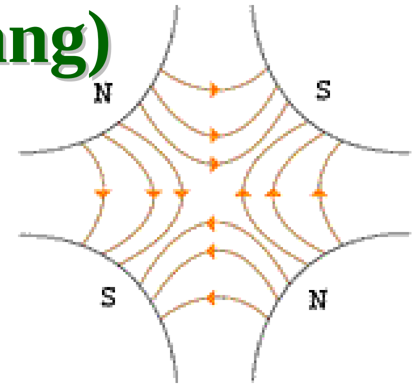
- By increasing *E* (hence *p*) and *B* together in a synchrotron, it is possible to maintain a constant radius and accelerate a beam of particles.
- In a synchrotron, the confining magnetic field comes from a system of several magnetic dipoles forming a closed arc.
- Dipoles are mounted apart, separated by straight sections/vacuum chambers including equipment for focusing, acceleration, injection, extraction, collimation, experimental areas, vacuum pumps.



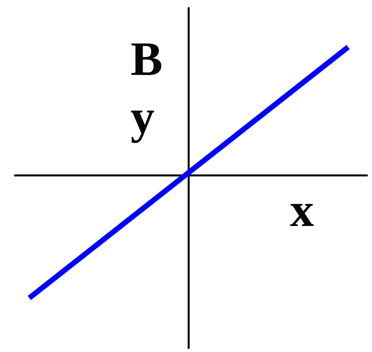


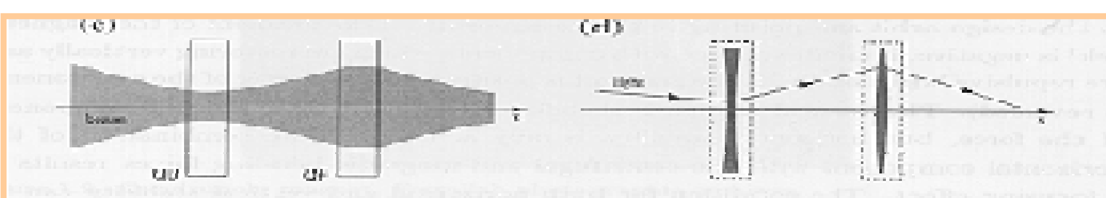
BEPCII dipole

2.3 Focusing (Prof. C.Ch. Wang)



- A sequence of focusing-defocusing fields provides a stronger net focusing force.
- Quadrupoles focus horizontally, defocus vertically or vice versa. Forces are linearly proportional to displacement from axis.
- A succession of opposed elements enable particles to follow stable trajectories, making small (betatron) oscillations about the design orbit.





BEPCII quadrupole

2.4 Hill's Equation

- **Equation of transverse motion**
  - ***Drift:*** $$x'' = 0, \quad y'' = 0$$
  - ***Quadrupole:*** $$x'' + k\,x = 0, \quad y'' - k\,y = 0$$
  - ***Dipole:*** $$x'' + \frac{1}{\rho^2}x = 0, \quad y'' = 0$$
  - ***Sextupole:*** $$x'' + k_s(x^2 - y^2) = 0, \quad y'' - 2k_s xy = 0$$
- **Hill's Equation:**

$$\boxed{x'' + k_x(s)x = 0, \quad y'' + k_y(s)y = 0}$$

Solution of the Hill's Equation

- Hill's equation (linear-periodic coefficients)

$$\frac{d^2u}{ds^2} + k(s)u = 0$$

where $k(s) = -\frac{1}{B\rho}\frac{dB_y}{dx}$

and u denotes x and y

- like restoring constant in harmonic motion, the solution is

$$u(s) = \sqrt{\varepsilon\beta(s)}\sin(\phi(s) + \phi_0)$$

- Condition $\phi(s) = \int \frac{ds}{\beta(s)}$

- Property of machine

$\beta(s)$ – envelope function

- Property of the beam

ε – Emittance

- Physical meaning (H or V planes)
  - Envelope

$$u_{\mathrm{env}}(s) = \sqrt{\varepsilon\beta(s)}$$

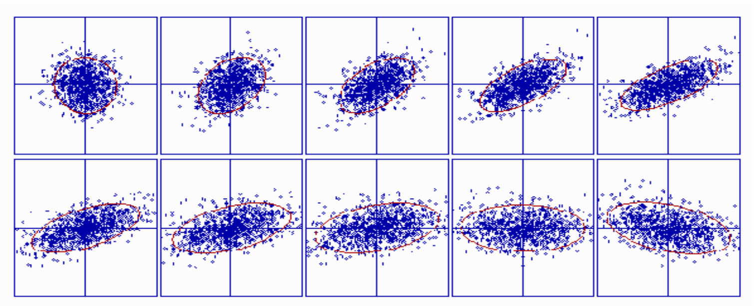
  - Maximum excursions

$$u'(s) = \sqrt{\varepsilon / \beta(s)}$$

2.5 Phase Space

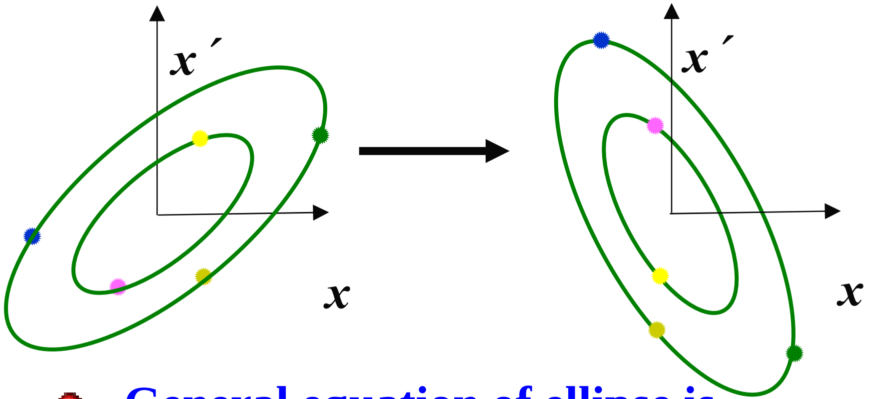
- **Under linear forces, any particle moves on an ellipse in phase space (x, x').**
- **Ellipse rotates in magnets and shears between magnets, but its area is preserved:**

Emittance



- ***Acceptance: $A_{x,y} > \varepsilon_{x,y}$***

$$A_x = \left(\frac{X^2}{\beta_x}\right)_{\max}, \quad A_y = \left(\frac{Y^2}{\beta_y}\right)_{\max}$$



- **General equation of ellipse is**

$$\beta x'^2 + 2\alpha x x' + \gamma x^2 = \varepsilon$$

- **α, β, γ are functions of distance (Twiss parameters), and ε is a constant. Area = πε.**
- **For non-linear beams can use 95% emittance ellipse or RMS emittance**

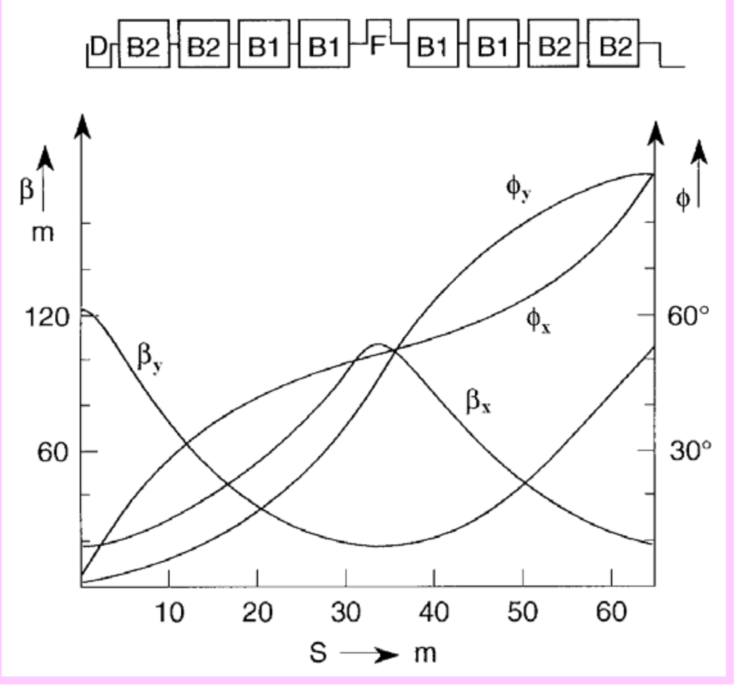
$$\varepsilon_{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

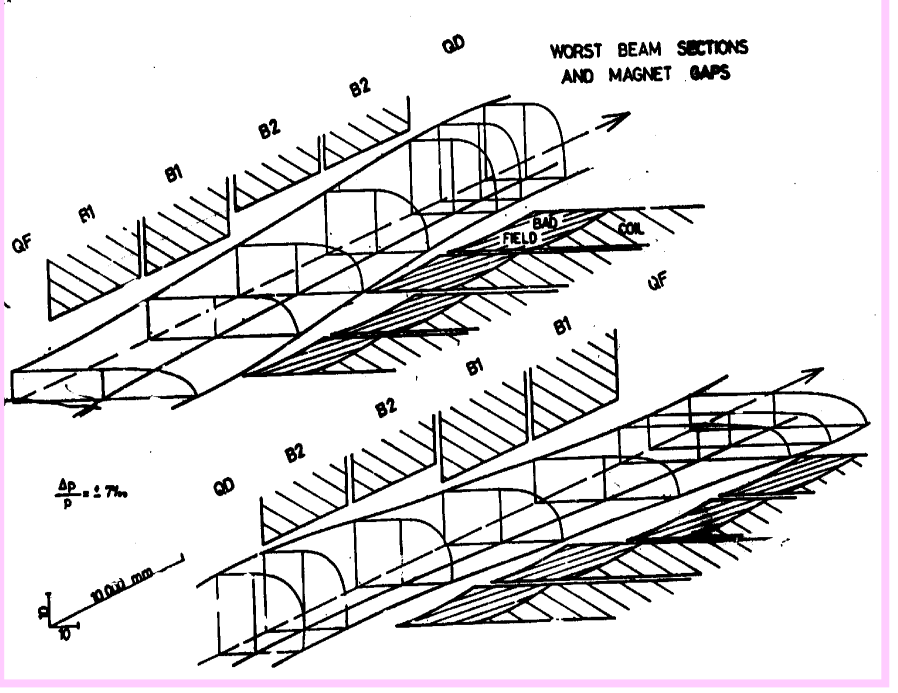
(statistical definition)

2.6 Lattice

(Prof. C.C. Kuo)

- **The pattern of focusing magnets, bending magnets and straight sections connecting in between;**
- **It has a strong influence on aperture of vacuum chambers and thus other systems of the accelerator.**





Matrix formalism

- As a consequence of the linearity of Hill's equations, we can describe the evolution of the trajectories in a lattice by means of linear transformations

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C(s) & C'(s) \\ S(s) & S'(s) \end{pmatrix} \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix} = M(s_2/s_1) \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

- In terms of the amplitude and phase function the transfer matrix is

$$M(s_2/s_1) = \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}(\cos\Delta\phi + \alpha_0 \sin\Delta\phi) & \sqrt{\beta(s)\beta_0}\sin\Delta\phi \\ -\dfrac{(\alpha(s)-\alpha_0)\cos\Delta\phi + (1+\alpha(s)\alpha_0)\sin\Delta\phi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta(s)}}[\cos\Delta\phi - \alpha(s)\sin\Delta\phi] \end{pmatrix}$$

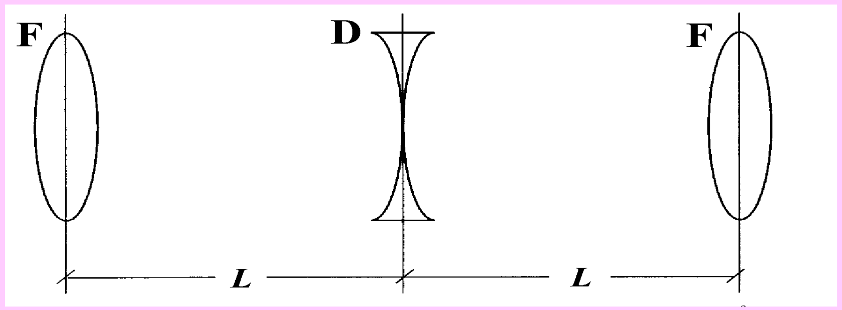
where $\alpha(s) = d\beta/ds$.

- For a periodic machine the transfer matrix over a turn reduces to

$$M(s_0/s_0) = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix} \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

Example: FODO Lattice

- The matrix for one period between mid-planes of F magnet in thin lens approximation is



$$M_{x,y} = \begin{pmatrix} 1 & 0 \\ \mp 1/2f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mp 1/2f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L^2/2f^2 & 2L(1 \pm L/2f \\ -L/2f^2(1 \mp L/2f) & 1 - L^2/2f^2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ \gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

$$\mu_{x,y} = \cos^{-1}(1 - \frac{L^2}{2f^2})$$

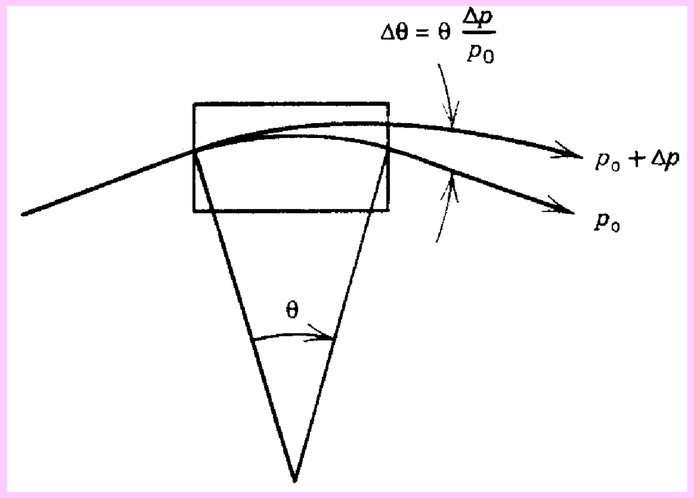
$$\beta_{x,y} = 2L \frac{1 \pm \sin(\mu/2)}{\sin\ \mu} = \frac{1}{\gamma_{x,y}}$$

$$\alpha_{x,y} = 0$$

$$\nu = (N_c \mu)/2\pi$$

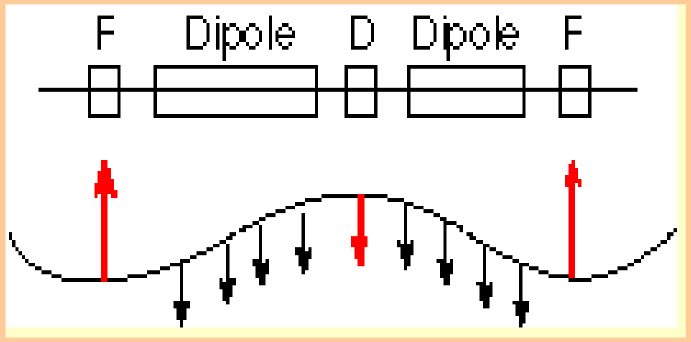
2.7 Dispersion

- **The dispersion has its origin that a particle of higher momentum is deflected through a lesser angle in a bending magnet.**



- **It was shown that the equations of motion of a charged particle is a linear Hill's equation**

$$x'' + \left(\frac{1}{\rho^2} - k(s)\right)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$



- **The solutions of the equation can be written in terms of the optics functions**

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\phi(s) - \phi_0) + \frac{\Delta p}{p_0} D(s)$$

$$D = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu} \oint \frac{\sqrt{\beta(\bar{s})}}{\rho(\bar{s})} \cos[\pi\nu - \phi(\bar{s}) - \phi(s)] d\bar{s}$$

2.8 Orbit Distortion

- Dipole errors may cause orbit distortion of particle beam.

Element	Source	Field Error	Deflection	Direction
Drift space (l=d)	Stray field	$\langle\Delta B_s\rangle$	$\langle\Delta B_s\rangle/B\rho$	x, y
Dipole (angle=θ)	Field error	$\langle\Delta B/B\rangle$	$\theta\langle\Delta B/B\rangle$	x
	Tilt	$\langle\Delta\theta_z\rangle$	$\theta\langle\Delta\theta_z\rangle$	y
Quadrupole (Kl)	Displacement	$\langle\Delta x,y\rangle$	$Kl\langle\Delta x,y\rangle$	x, y

- Similar to dispersion case, the equation of motion is written as

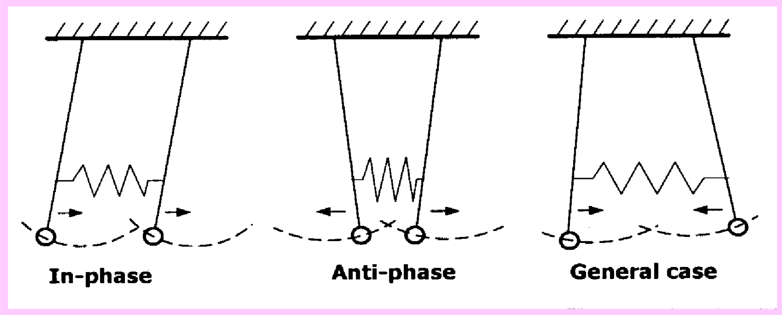
$$y''+ky=\frac{1}{B\rho}\frac{\Delta B(s)}{B}$$

- The solutions are $$y(s)=\sqrt{\varepsilon\beta(s)}\cos(\phi(s)-\phi_0)+y_{cod}(s)$$

$$y_{COD}=\frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\oint\frac{\Delta B(s)\sqrt{\beta(\bar{s})}}{B\rho(\bar{s})}\cos[\pi\nu-\phi(\bar{s})-\phi(s)]d\bar{s}$$

2.9 Linear Coupling

- **Coupling is the phenomena that energy exchange between two oscillators.**

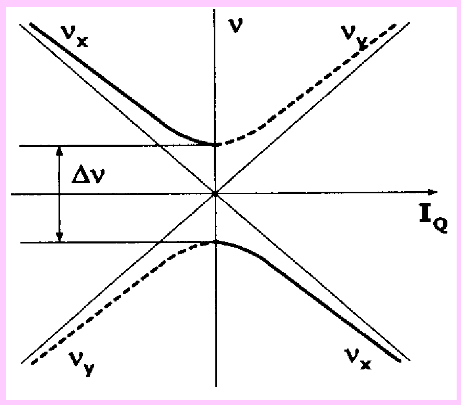


- **In accelerators, the horizontal and vertical motions are also coupled:**

$$x'' + k_x x = -(k + M'/2)y - My'$$

$$y'' + k_y y = -(k - M'/2)x - Mx'$$

Where $k = -\dfrac{1}{2B\rho}\left(\dfrac{\partial B_x}{\partial x} - \dfrac{\partial B_y}{\partial y}\right)_0, \quad M = -\dfrac{B_s}{B\rho}$



- ***x-y* coupling can be compensated with skew quadrupoles and anti-solenoids.**
- **Measured with tune approaching.**

$$\kappa = \frac{\varepsilon_y}{\varepsilon_x} = \frac{(\Delta\nu)^2}{(\nu_x - \nu_y)^2 + 2(\Delta\nu)^2}$$

2.10 Non-linearity

- **Hill's equation with higher order terms in the expansion of magnetic field**

$$B_z + iB_x = B_0\rho_0\left[\sum_{n=1}^{M}\frac{k_n(s)+ij_n(s)}{n!}(x+iz)^n\right]$$

$$k_n = \frac{1}{B_0\rho_0}\left.\frac{\partial^n B_y}{\partial x^n}\right|_{(0,0)}$$ **Normal multipoles**

$$j_n = \frac{1}{B_0\rho_0}\left.\frac{\partial^n B_x}{\partial x^n}\right|_{(0,0)}$$ **Skew multipoles**

- **The Hill's equations acquire additional nonlinear terms:**

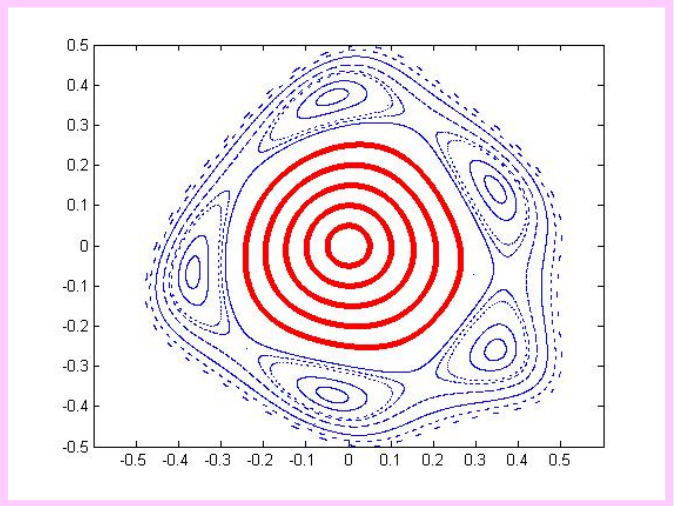
$$\frac{d^2x}{ds^2}+\left(\frac{1}{\rho^2(s)}-k_1(s)\right)x = \mathrm{Re}\left[\sum_{n=2}^{M}\frac{k_n(s)+ij_n(s)}{n!}(x+iz)^n\right]$$

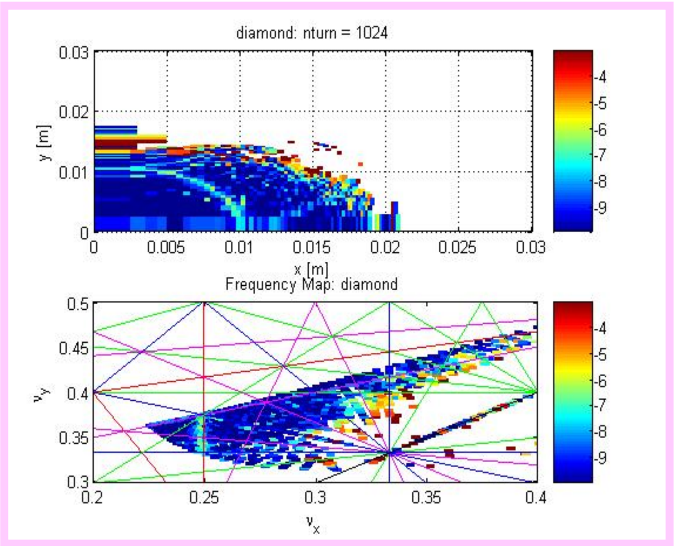
$$\frac{d^2z}{ds^2}+k_1(s)z = -\mathrm{Im}\left[\sum_{n=2}^{M}\frac{k_n(s)+ij_n(s)}{n!}(x+iz)^n\right]$$

- **There is no analytical solution available in general, and the equations have to be solved by tracking or perturbative analysis.**

Analytical and Tracking Studies

- Stable and unstable fixed points appears which are connected by separatrices.
- Islands enclose the stable fixed points.
- On a resonance the particle jumps from one island to next and the tune is locked at the resonance value.
- Region of chaotic motion appear.
- The region of stable motion, called dynamic aperture, is limited by the appearance of unstable fixed points.





3. Longitudinal Motion

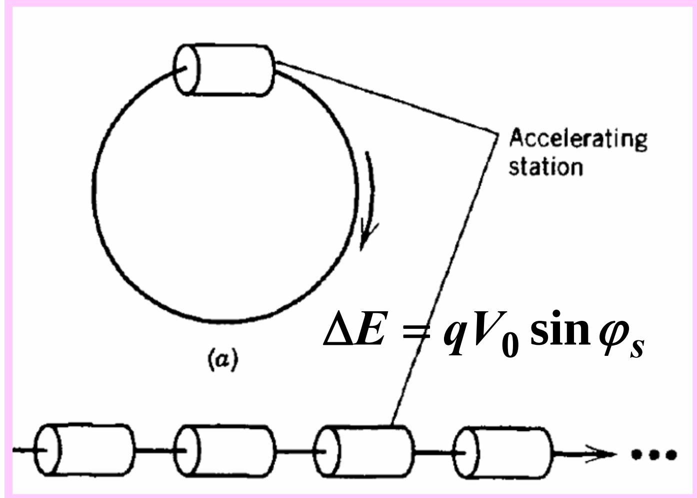
(Prof. S.Y.Lee)

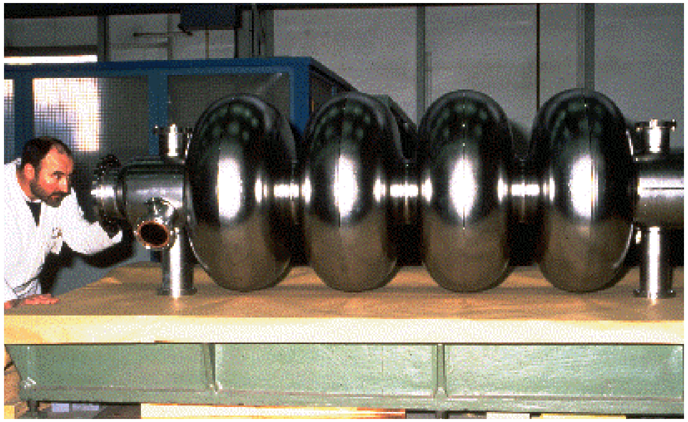
- **RF Cavities & Acceleration**
- **Synchrotron Oscillation**
- **Bunches and Buckets**
- **γ-Transition**

3.1 RF Cavities & Acceleration

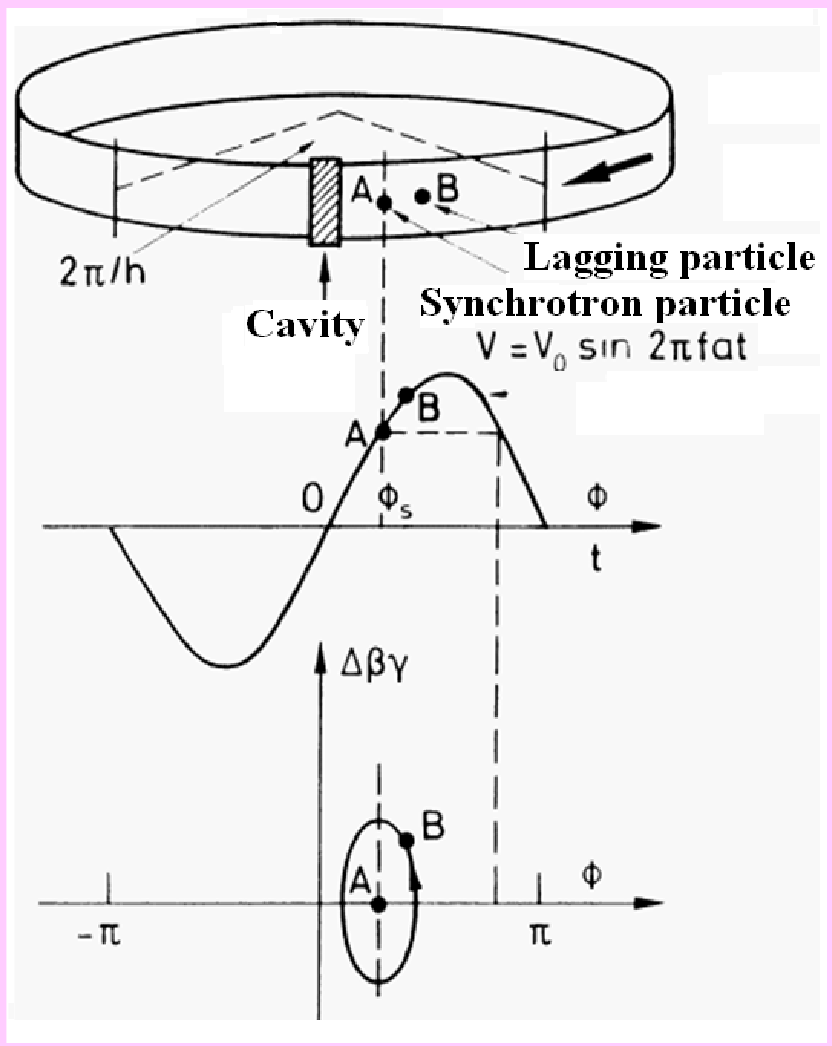
(Prof. R.F. Wang)

- Necessary conditions for acceleration.
- Both linear and circular accelerators use electromagnetic fields oscillating in resonant cavities to apply the accelerating force.
- Linac– particles follow straight path through series of cavities.
- Circular accelerators – particles follow circular path in *B*-field and particles return to same accelerating cavity each time around.





3.2 Synchrotron Oscillation



- This is a biased rigid pendulum

$$\ddot{\phi} = -\frac{2\pi V_0 h \eta f_0^{\,2}}{E_0 \beta^2 \gamma}\left(\sin\phi - \sin\phi_s\right)$$

- For small amplitudes

$$\ddot{\phi} + \frac{2\pi V_0 h \eta f_0^{\,2}}{E_0 \beta^2 \gamma}\Delta\phi = \ddot{\phi} + \Omega_s^{\,2}\Delta\phi = 0$$

- Synchrotron frequency

$$f_s = \sqrt{\frac{\eta h V_0 \cos\phi_s}{2\pi E_0 \beta^2 \gamma}}\, f_0$$

- Synchrotron tune

$$\nu_s = \frac{f_s}{f} = \sqrt{\frac{\eta h V_0 \cos\phi_s}{2\pi E_0 \beta^2 \gamma}}$$

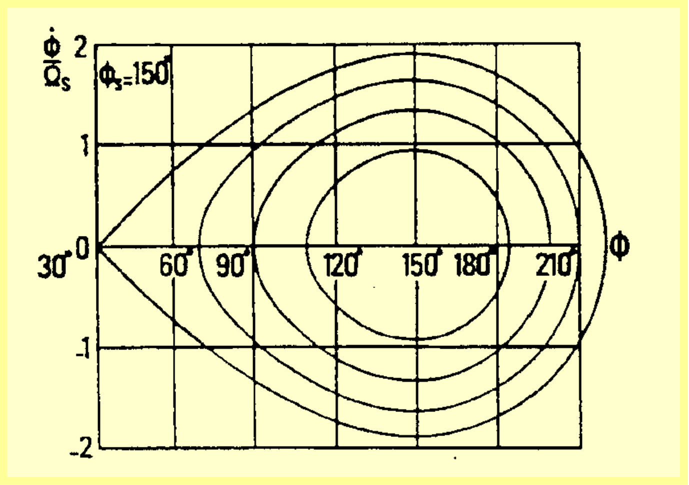
3.3 Bunches and Buckets

- **For large amplitudes**

$$\ddot{\phi} = -\frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right)$$

- **Integrated becomes an invariant**

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = const.$$



- **The second term is the potential energy function, and the equation of each separatrix is**

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left[\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right].$$

- **And the half height when** $\ddot{\phi} = 0$ **is**

$$\left(\Delta E/E_s\right)_{\max} = \pm\beta\left\{\frac{eV_0}{\pi h\eta E_s}G(\phi_s)\right\}^{1/2} \qquad G(\phi_s) = \left[2\cos\phi_s - (\pi - 2\phi_s)\sin\phi_s\right]$$

- **Bucket size should be larger than bunch size ($\Delta E/E_s$ and $\Delta\phi$ ($\Delta\tau$, Δz))**

3.4 γ-Transition

- **Recall synchrotron focusing strength:** $$\Omega_s^2 = \frac{2\pi\eta h V_0 \cos\phi_s f_0^2}{E_0\beta^2\gamma}$$

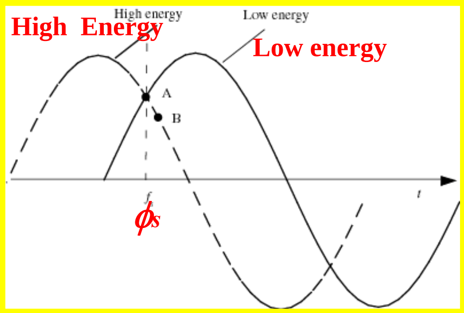
- **If $\eta\cdot\cos\phi_s<0$, then $\Omega^2<0$, the oscillation gets unstable, where**

$$\eta = \frac{df/f}{dp/p} = \frac{p}{\beta}\frac{d\beta}{dp} + \frac{p}{R}\frac{dR}{dp} = \frac{1}{\gamma^2} - \frac{\overline{D}}{R_0} \qquad R = R_o + D\cdot\frac{\Delta p}{p}$$

- **The η changes from positive to negative at transition γ:**

$$\eta = \frac{1}{\gamma^2} - \frac{\overline{D}}{R_0} = 0 \quad \text{or} \quad \gamma_{tr} = \sqrt{\frac{R_0}{\overline{D}}}$$

- **The motion is stable for $\phi_s\in(0, \pi/2)$, and unstable $\phi_s\in(0, \pi)$ for $\gamma<\gamma_{tr}$, and vice versa for $\gamma>\gamma_{tr}$.**



- **To cross γ_{tr} should be avoided in design or apply the phase-jump technique.**

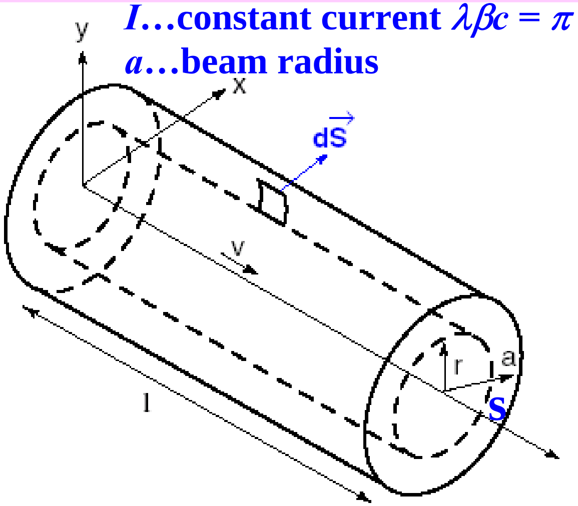
4. Collective Effects

(Prof. A.W Chao)

- **Space Charge**
- **Wake Fields and Impedance**
- **Coherent Instabilities**
- **Beam-beam Effects**
- **Landau Damping**

4.1 Space Charge

η…charge density in C/m³
λ… constant line charge $\pi a^2 \eta$
I…constant current $\lambda\beta c = \pi a^2 \eta\beta c$
a…beam radius



Electric

$$\vec{E} = E_r$$

$$\operatorname{div} \vec{E} = \frac{\eta}{\varepsilon_0}$$

Magnetic

$$\vec{B} = B_\varphi$$

$$\operatorname{curl} \vec{B} = \mu_0 \vec{J}$$

Current density ($\beta c \eta$)

$$\iiint \operatorname{div} \vec{E}\, dV = \iint \vec{E}\, d\vec{S} \qquad \oint \vec{B}\, r\, d\vec{\varphi} = \iint curl\, \vec{B}\, d\vec{A}$$

Volume element

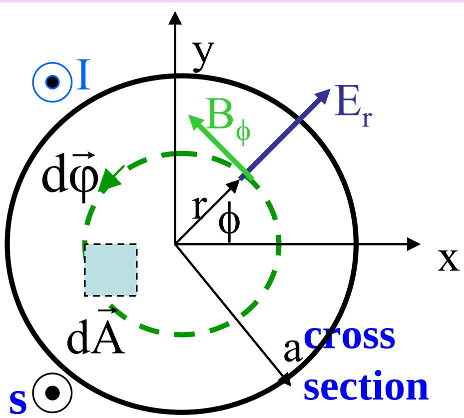
Apply these integrals over

cylinder radius r
length l

cross section
radius r

$$r^2 \pi l \frac{\eta}{\varepsilon_0} = E_r 2r\pi\pi \qquad B_\varphi 2r\pi = \mu_0 r^2 \pi\beta c\eta$$

$$\boxed{E_r = \frac{I}{2\pi\varepsilon_0 \beta c} \frac{r}{a^2}} \qquad \boxed{B_\varphi = \frac{I}{2\pi\varepsilon_0 c^2} \frac{r}{a^2}}$$



For $\beta \rightarrow 1$ ($\gamma >> 1$)

$$F_E = qE_r = -F_B = -q \cdot B_\phi \cdot v \quad \text{so} \quad \Sigma F = 0$$

Incoherent Tune Shift in a Synchrotron

- **Beam not bunched (so no acceleration)**
  Uniform density in the circular *x*-*y* cross section (not very realistic)

$$x'' + \left(K(s) + K_{SC}(s)\right)x = 0$$

➔ ν_{x0} **(external) + $\Delta\nu_x$ (space charge)**

For small "gradient errors" k_x

$$\Delta\nu_x = \frac{1}{4\pi}\int_0^{2\pi R} k_x(s)\beta_x(s)ds = \frac{1}{4\pi}\int_0^{2\pi R} K_{SC}(s)\beta_x(s)ds$$

$$\Delta\nu_x = -\frac{1}{4\pi}\int_0^{2\pi R}\frac{2r_0 I}{e\beta^3\gamma^3 c}\frac{\beta_x(s)}{a^2}ds = -\frac{r_0 RI}{e\beta^3\gamma^3 c}\left\langle\frac{\beta_x(s)}{a^2(s)}\right\rangle = -\frac{r_0 RI}{e\beta^3\gamma^3 c\varepsilon_x}$$

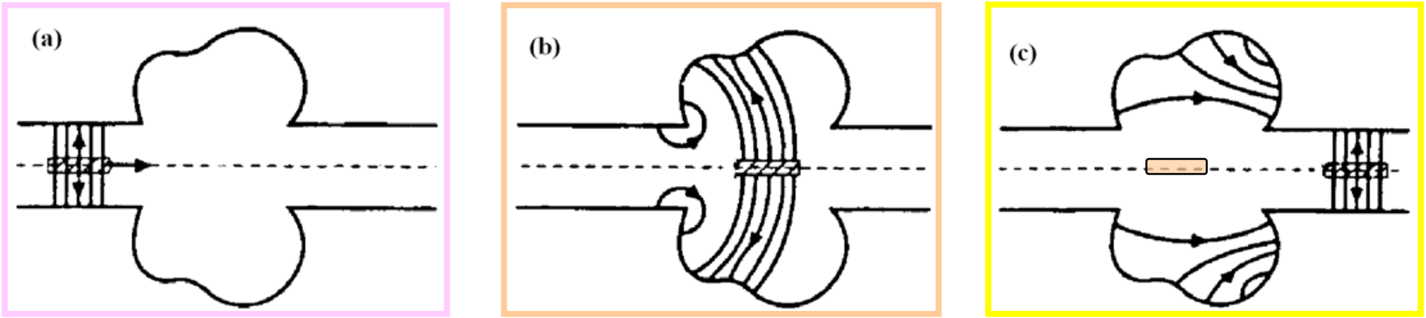
$$\boxed{\Delta\nu_{x,y} = -\frac{r_0 N}{2\pi\varepsilon_{x,y}\beta^2\gamma^3}}$$

Using $I = (Ne\beta c)/(2\pi R)$ with
N…number of particles in ring
$\varepsilon_{x,y}$ are x, y emittance containing 100% of particles

- **"Direct" space charge, unbunched beam in a synchrotron**
  - **Vanishes for $\gamma >> 1$**
  - **Important for low-energy machines**
  - **Independent of machine size $2\pi R$ for a given N**

4.2 Wake Fields and Impedance

- Wake field will be driven by a beam when there is a discontinuity in the vacuum chamber:



$$\vec{F}_{\perp}(r,\theta,z) = -eI_m W_m(z) m r^{m-1}(\hat{r}\cos m\theta - \hat{\theta}\sin m\theta)$$

$$F_{\|}(r,\theta,z) = -eI_m W'_m(z) r^m \cos m\theta$$

- Impedances are just Fourier transforms of wake functions:

$$Z_m^{\perp}(\omega) = \frac{i}{v/c}\int \frac{dz}{v} e^{-i\omega z/v} W_m(z) \quad , \quad Z_m^{\|}(\omega) = \int_{-\infty}^{\infty} \frac{dz}{v} e^{-i\omega z/v} W'_m(z)$$

Impedance $Z = Z_r + iZ_i$

Induced voltage $V \sim I_w Z = -I_B Z$

V acts back on the beam

⇨ Intensity dependant

4.3 Instabilities *Linacs*

- **Energy variation along the bunch length**
  - **Induced by longitudinal wake fields**
  - **It can be compensated for by properly phasing.**
- **Transverse beam break-up**
  - **Induced by transverse wake fields**

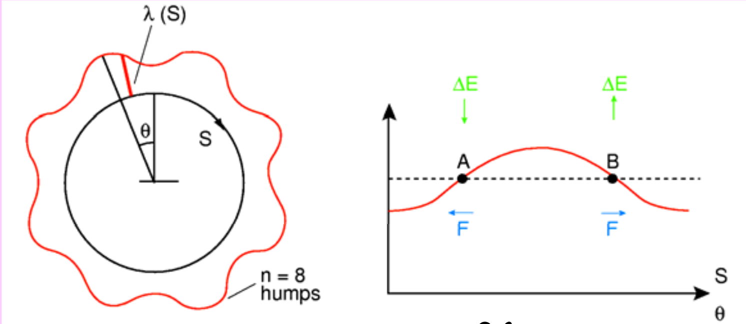
$$\frac{d}{ds}\left[E(s)\frac{d}{ds}x(z,s)\right]+E(s)k^2(s)x(z,s)=\frac{Ne^2}{L}\int_z^{\infty}\lambda(z')W_1(z'-z)x(z',s)dz'$$

  - **It can be cured by the BNS damping: applying stronger focusing on tail than head of the bunch to balance the wake.**

$$\frac{\Delta k}{k}\approx-\frac{Ne^2W_1(l)}{4k^2LE_f}\ln\frac{E_f}{E_i}$$

Instabilities — *Circular accelerators*

- **Negative Mass Instability**

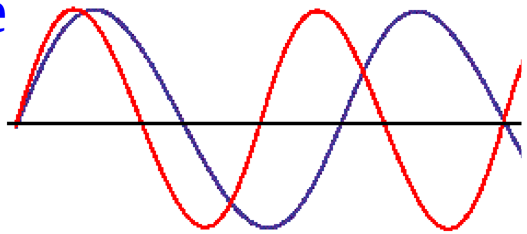


$$E = -\frac{e}{4\pi\varepsilon_0\gamma^2}\frac{\partial\lambda}{\partial s}$$

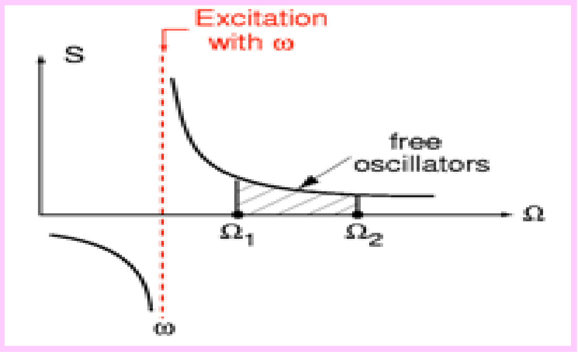
- **Robinson instability**
  - Caused by fundamental cavity mode
  - Beam will be stabilized by properly tuning the cavity ($h\omega_0 > \omega_{rf}$ for $\gamma > \gamma_t$).
- **Head-tail Instability**
  - Caused by transverse wake field
  - Beam will be stabilized by properly setting chromaticity ($\xi > 0$ for $\gamma > \gamma_t$).
- **Strong head-tail Instability**
  - Caused by transverse wake as counterpart of beam break-up in a linac.
  - Once the threshold is exceeded the instability tends to grow very fast.
- **Microwave instability** – *increasing energy spread and bunch length*
- **Potential-well distortion** – *increasing bunch length*
- **Coupled bunch instability**
  - multi-bunch effects caused by long-range transverse & longitudinal wakes
  - Cure: control of impedance and applying feedback systems

4.4 Landau Damping

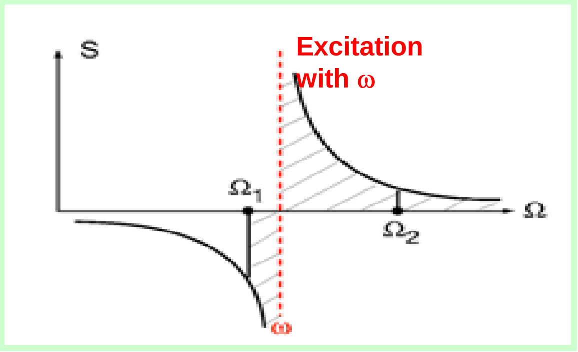
- Two oscillators excited together become incoherent and give zero centre of charge motion after a number of turns comparable to the reciprocal of their frequency difference.



- External excitation is outside the frequency range of the oscillators: No damping



- External excitation is inside the frequency range of the oscillators The integral has a pole at $\Omega = \omega$: Landau damping

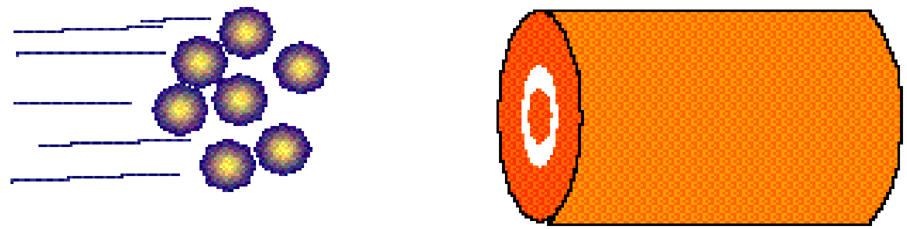


4.5 Beam-beam Effects

W = Energy available in center-of-mass for making new particles

For fixed target :

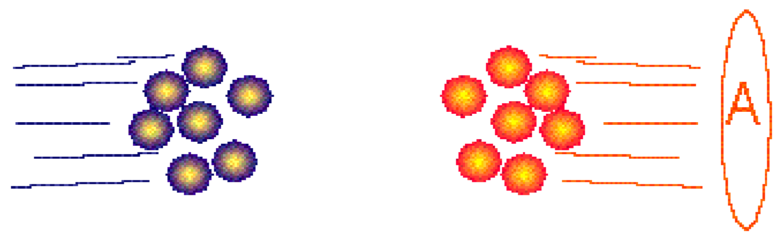
$$E_{c.m.} \cong \sqrt{2 m_T E_B}$$



... and we rapidly run out of money trying to gain a factor 10 in c.m. energy

But a storage ring , colliding two beams, gives:

$$E_{c.m.} \cong 2 \; E_B$$



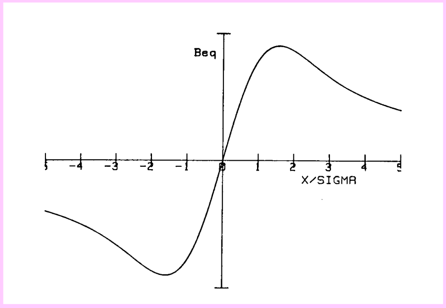
Problem: Smaller probability that accelerated particles collide .... "Luminosity" of a collider

$$L = N_1 N_2 \frac{1}{A} \frac{\beta c}{2\pi R} \approx 10^{29} \ldots 10^{34} \, cm^{-2} s^{-1}$$

- **Beam-beam interactions are non-linear and collective effects**

$$\rho(r) = \frac{ne}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \qquad F_r = -\frac{ne^2}{2\pi r \varepsilon_0}\left(1 \pm \beta^2\right)\left(1 - e^{-r^2/2\sigma^2}\right)$$

- **For small amplitude particles, the beam-beam force can be approximated as a focusing lens in both x and y planes with focal length *f* given by**



$$(f_{x,y})^{-1} = \frac{2Nr_e}{\gamma\sigma_{x,y}(\sigma_x + \sigma_y)}$$

- **The incoherent linear beam-beam tune shifts are**

$$\Delta\nu_{x,y} = \left(f_{x,y}\right)^{-1} \frac{\beta^*_{x,y}}{4\pi} = \frac{Nr_e\beta^*_{x,y}}{2\pi\gamma\sigma_{x,y}(\sigma_x + \sigma_y)} = \xi_{x,y}$$

- **With the maximum beam-beam tune shift, luminosity is**

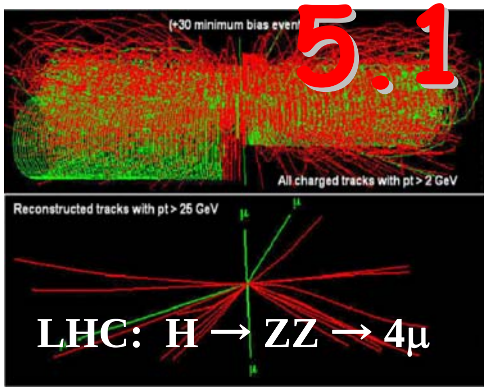
$$L = \frac{\pi f_c \varepsilon_x \xi_x \xi_y (1 + R)^2}{r_0^2 \beta^*_y}$$

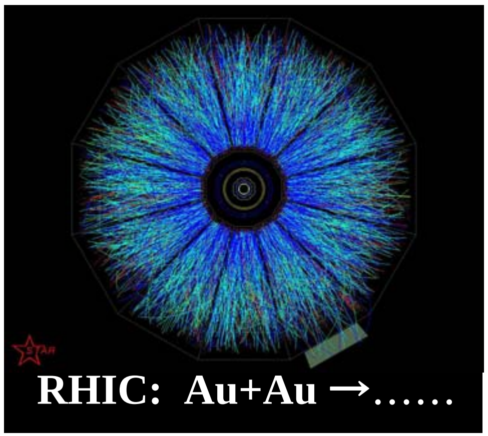
- **Exhaustive study can be carried out with beam-beam simulation.**

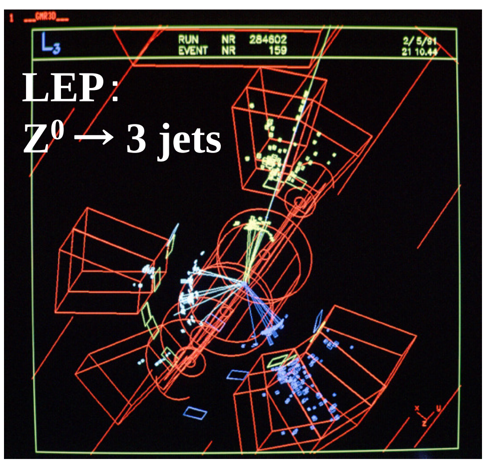
5. Leptons and Hadrons

- Leptons vs. Hadrons
- Synchrotron radiation
- Radiation damping
- Radiation excitation
- Rings vs. Linacs

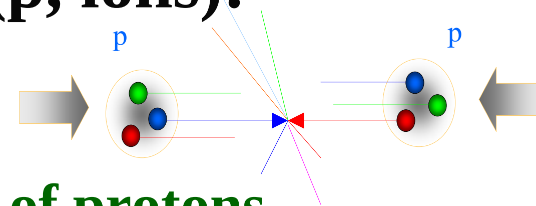
5.1 Leptons vs. Hadrons







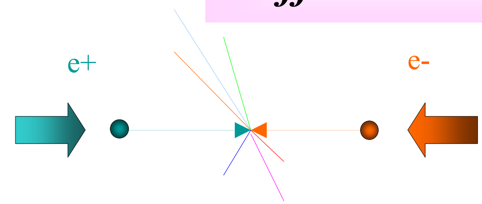
- **Hadron Collider (p, ions):**



  - **Composite nature of protons**
  - **Can only use p_t conservation**
  - **Huge QCD background**

$$E_{eff}^{e^+e^-} \approx 10 \cdot E_{eff}^{pp}$$

- **Lepton Collider:**



  - **Elementary particles**
  - **Well defined initial state**
  - **Beam polarization**
  - **Produces particles democratically**
  - **Momentum conservation eases decay product analysis**

5.2 Synchrotron radiation

- The electromagnetic radiation emitted when the charged particles are accelerated;

- The particle loses the energy of U_0 in one turn;

$$U_0 = \frac{e^2\gamma^4}{3\varepsilon_0\rho}$$

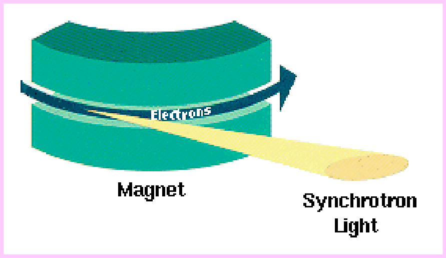
- For electrons:

$$U_0(keV) = 88.46\frac{E(GeV)^4}{\rho(m)}$$

e.g. LEP2: E=100GeV, ρ=3100 m, U_0=1.85 GeV

- For protons:

$$U_0(keV) = 7.78\times10^{-12}\frac{E(GeV)^4}{\rho(m)}$$

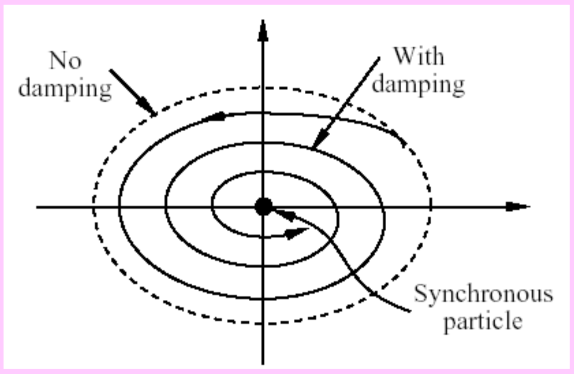


e.g. LHC: E=7000GeV, ρ= 2804 m, U_0= 6.66 keV

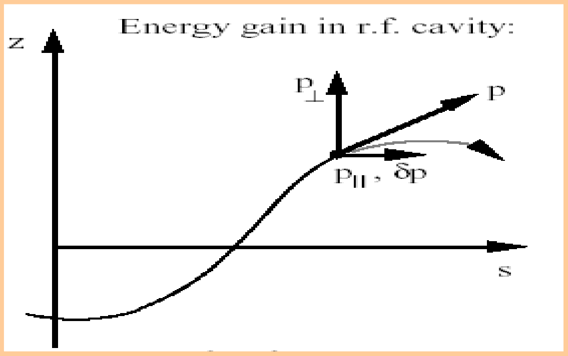
5.3 Radiation damping

- Synchrotron radiation loss is compensated by the RF fields:
- This will cause radiation damping.

$$\frac{d^2\varepsilon}{dt^2}+\frac{2}{\tau_s}\frac{d\varepsilon}{dt}+\omega_s^2\varepsilon=0$$

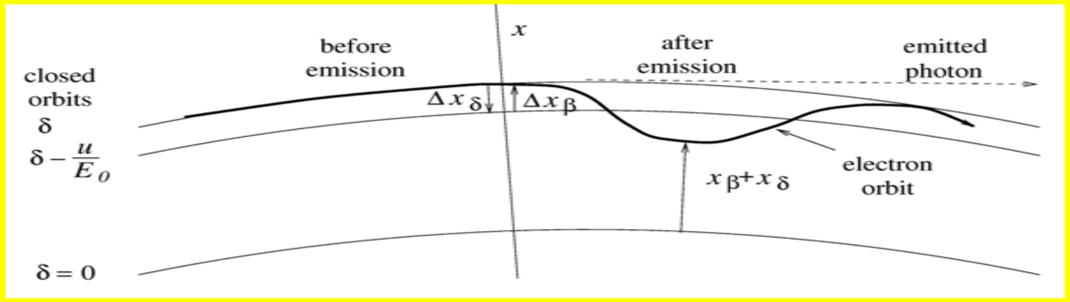


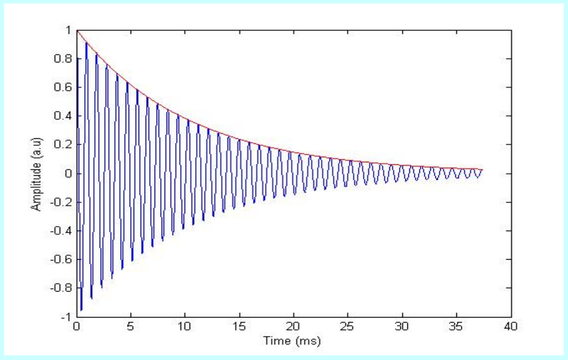
- Longitudinal: higher the energy of particle more SR loses;
- Vertical: radiated momentum is with $P_\perp$, while RF compensates $P_{||}$;



- Horizontal: After the photon emission

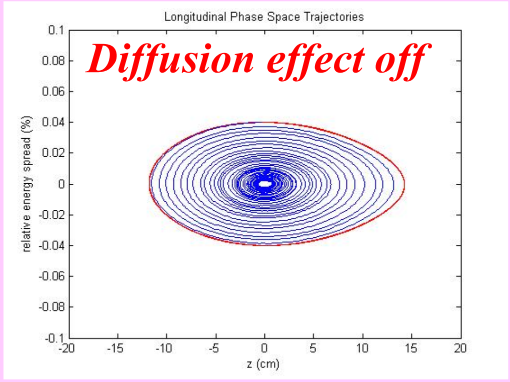
$$\delta x_\beta+\delta x_\varepsilon=0$$

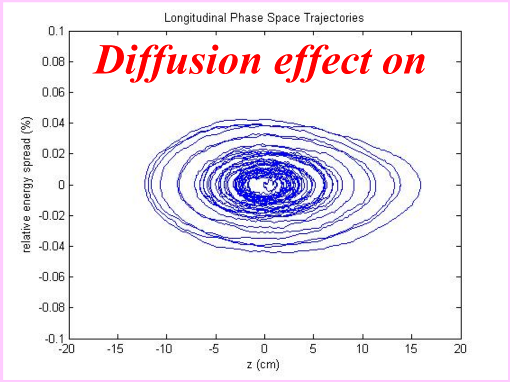




5.4 Radiation excitation

- With pure damping, the emittance will be zero in *x*, *y* and *z* planes.
- The radiated energy is emitted in quanta: each quantum carries an energy $u = \hbar\omega$;
- The emission process is instantaneous and the time of emission of individual quanta is statistically independent;
- Radiation damping combined with radiation excitation determine the equilibrium beam distribution and therefore emittance, beam size, energy spread and bunch length.





$$\sigma_E = \left(\frac{55}{64\sqrt{3}} \frac{\hbar}{\rho mc} \right)^{1/2} \cdot \gamma \qquad \sigma_z = \frac{\alpha c}{\Omega_s} \sigma_E$$

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{J_x} \frac{\langle H/\rho^3 \rangle}{\langle 1/\rho^2 \rangle} \qquad \varepsilon_y = \frac{55}{64\sqrt{3}} \frac{\hbar}{mc} \frac{<\beta_y/\rho^3>}{J_y <1/\rho^2>}$$

J_i are damping partition numbers, H is dispersion invariant $H(s) = \gamma D_x^2 + 2\alpha D_x D_x' + \beta D_x'^2$

5.5 Rings vs. Linacs *(electrons)*

Rings

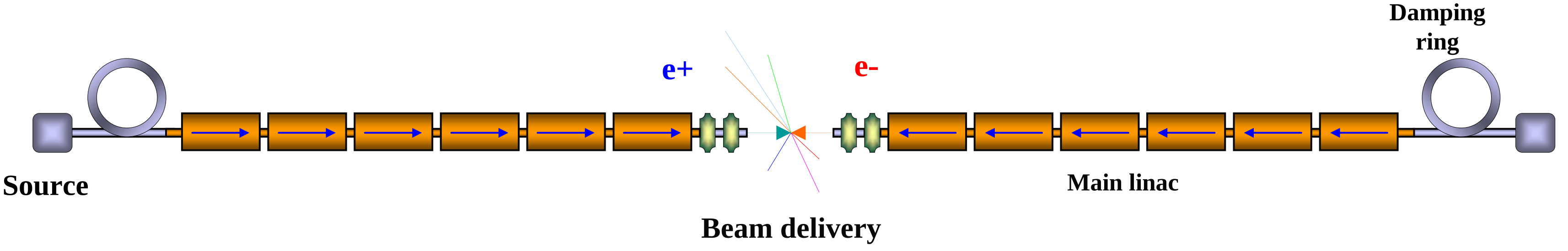
- Accelerate + collide every turn
- 'Re-use' RF + 're-use' particles

  $\Rightarrow$ efficient
- Finite beam emittance.
- Cost:
  - Linear costs(magnets, tunnel, etc.) : $\$_{lin} \propto \rho$
  - RF costs: $\$_{RF} \propto U_0 \propto E^4/\rho$
  - Optimum when: $\$_{lin} \propto \$_{RF} \Rightarrow \rho \propto E^2$
  - The size and the optimized cost scale as E^2
  - Also the energy loss per turn scales as E^2

  $\Rightarrow$ get inefficient when $E\uparrow\uparrow$

Linacs

- One-pass acceleration + collision $\Rightarrow$ need high gradient
- NO bending magnets $\Rightarrow$ NO synchrotron radiation
- Small beam size to reach a high Luminosity
- Much less limited by beam-beam effect
- A lot of accelerating structures
- Cost scaling linear with E



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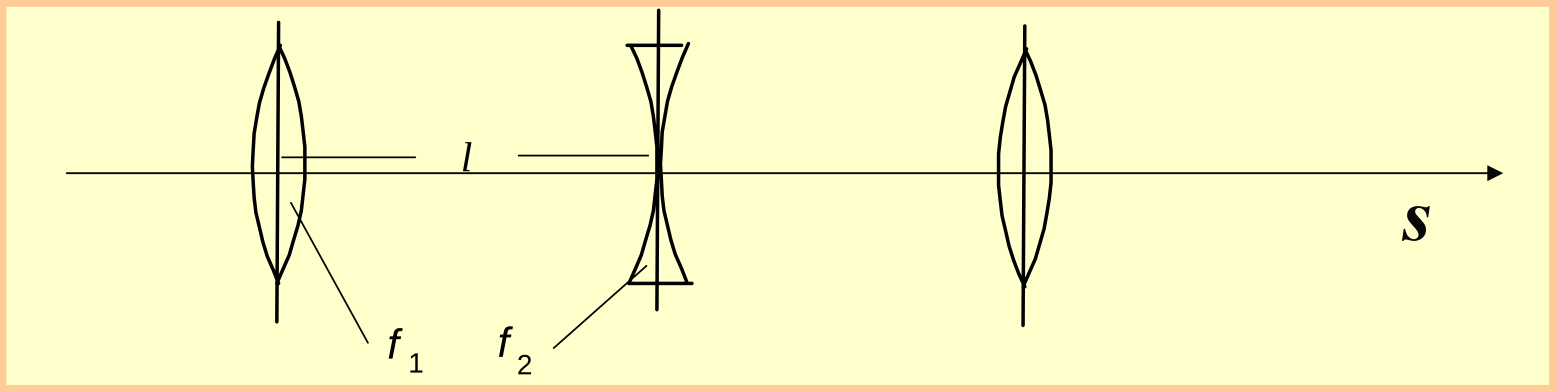
Questions

(1) A 50 GeV (kinetic energy) proton synchrotron will receive an injected beam at 4 GeV (kinetic energy) from an injector. The 2 T magnetic field of the bending magnets is excited at top energy of 50GeV. Given that the rest mass of the proton E_0 is 0.93827 GeV:

a) What is the $B\rho$ at 4 GeV and at 50 GeV?

b) What is the bending radius ρ?

(2) A quadrupole doublet (half of the FODO cell from the mid-point of the F to the mid point D) consists of two lenses of focal length f_1 and f_2 separated by a drift length of l.



Please show, by writing the three matrices for the lenses, the product matrix is:

$$M = \begin{pmatrix} 1 - \dfrac{l}{f_1} & l \\ -\dfrac{1}{f^*} & 1 + \dfrac{l}{f_2} \end{pmatrix}$$

where $$\frac{1}{f^*} = \frac{1}{f_1} - \frac{1}{f_2} + \frac{l}{f_1 f_2}$$

Thank You